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KNOTS WITH BRAID INDEX THREE HAVE PROPERTY-P

W. MENASCO* and X. ZHANG†,‡
Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260-2900, USA
*menasco@math.buffalo.edu
†xinzhang@math.buffalo.edu

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ABSTRACT
We prove that knots with braid index three in the 3-sphere satisfy the Property P conjecture.

1. Introduction

Let K be a knot in the 3-sphere S^3 and $M = M_K$ the complement of an open regular neighborhood of K in S^3. As usual, the set of slopes on the torus ∂M (i.e. the set of isotopy classes of unoriented essential simple loops on ∂M) is parameterized by

$$\{m/n\,;\ m, n \in \mathbb{Z}, n > 0, (m, n) = 1\} \cup \{1/0\},$$

so that $1/0$ is the meridian slope of K and $0/1$ is the longitude slope of K. The manifold obtained by Dehn surgery on S^3 along the knot K (equivalently, Dehn filling on M along the torus ∂M) with slope m/n, is denoted by $K(m/n)$ or $M(m/n)$. Of course $K(1/0) = S^3$, and thus the surgery with the slope $1/0$ is called the trivial surgery. The celebrated *Property P* conjecture, introduced by Bing and Matin in 1971 [2], states that every nontrivial knot K in S^3 has Property P, i.e. every nontrivial surgery on S^3 along K produces a non-simply connected manifold. For convenience we say that a class of knots in S^3 have Property P if every nontrivial knot in this class has Property P. The following classes of knots were known to have Property P: torus knots [12], symmetric knots [5] (the part for strongly invertible knots was proved in [4]), satellite knots [8], arborescent knots [17], alternating knots [6], and small knots with no non-integral boundary slopes [7]. For a simple homological reason, to prove the conjecture for a knot K one only needs to consider the surgeries of K with slopes $1/n$, $n \neq 0$. A remarkable progress on the conjecture was made in [5]; it was proved there that for a nontrivial knot, only one of $K(1)$ or $K(-1)$ could possibly be a simply connected manifold. Another remarkable result

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was given in [11], which told us that if the Property P conjecture is false, then the Poincare conjecture is false. For some earlier progresses on the conjecture, see [13, Problem 1.15] for a summary. In this paper we prove

Theorem 1. *Knots in S^3 with braid index three satisfy the Property P conjecture.*

The main tools we shall use are the Casson invariant and essential laminations. We refer to [1] for the definition and basic properties of the Casson invariant and [10] for the definition and basic properties of an essential lamination. In Sec. 2 we give an outline of the proof of Theorem 1. Actually the proof of Theorem 1 is reduced there to that of two propositions, Proposition 3 and Proposition 4. These two propositions will then be proven in Secs. 3 and 4 respectively.

2. Proof of Theorem 1

Recall that the 3-braid group, B_3, has the following well known Artin presentation:

$$B_3 = < \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 >$$

where σ_1 and σ_2 are elementary 3-braids as shown in Fig. 1.

Fig. 1. σ_1 (the left figure) and σ_2 (the right figure).

If we let $a_1 = \sigma_1, a_2 = \sigma_2, a_3 = \sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^{-1}$, then B_3 also has the following presentation in generators a_1, a_2 and a_3 (see [18]):

$$B_3 = < a_1, a_2, a_3 \mid a_2a_1 = a_3a_2 = a_1a_3 > .$$

In this paper we shall always express a 3-braid as a word $w(a_1, a_2, a_3)$ in letters a_1, a_2, a_3. Such a word is called *positive* if the power of every letter in the word is positive. A positive word $w = a_{\tau_1} \cdots a_{\tau_k}$ is said to be in *non-decreasing order* (ND-order) if the array of its subscripts $(\tau_1, \ldots, \tau_k)$ satisfies

$$\tau_{j+1} = \tau_j \text{ or } \tau_{j+1} = \tau_j + 1 \text{ (mod 3 if } \tau_j + 1 = 4) \text{ for } j = 1, \ldots, k-1.$$

One can define *negative* word and *non-increasing order* (NI-order) similarly. Let P be the set of positive words in ND-order, let N be the set of negative words in NI-order, and let $\alpha = a_2a_1$. It is proven in [18] that for any 3-braid, there is a representative in its conjugacy class that is a shortest word in a_1, a_2, a_3 and is of the form

(i) a product of α^k and a word (maybe empty) in P for some non-negative integer k; or

(ii) a product of α^k and a word (maybe empty) in N for some non-positive integer k; or

(iii) a product of a word in P and a word in N,

where the meaning of *the shortest* is that the length, i.e. the number of letters of the representative is minimal among all representatives in the conjugacy class of the braid. Such representative of a 3-braid is said to be in *normal form.* We shall only need to show that if K is a nontrivial knot in S^3 which is the closure of a 3-braid in normal form (i) or (ii) or (iii), then it has Property P.

Recall that a word $w(a_1, a_2, a_3)$ is called *freely reduced* if no adjacent letters are inverse to each other, and is called *cyclically reduced* if it is freely reduced and the first letter and the last letter of the word are not inverse to each other. Given a word $w(a_1, a_2, a_3)$, one can combine all adjacent letters of the same subscript into a single power of the letter, called a *syllabus* of the word in that subscript. A word w is called *syllabus reduced* if it is expressed as a word in terms of syllabuses as

$$w = a_{\tau_1}^{m_1} a_{\tau_2}^{m_2} \cdots a_{\tau_k}^{m_k}$$

such that $a_{\tau_j} \neq a_{\tau_{j+1}}$ for $j = 1, \ldots, k-1$. A word is called *cyclically syllabus reduced* if it is syllabus reduced and its first and last syllabuses are in different subscripts.

Let P^* denote the set of all positive words in a_1, a_2, a_3 such that between any two syllabuses in a_3 both a_1 and a_2 occur. Obviously any 3-braid of norm form (i) is contained in P^*. Let β be a 3-braid in P^*. Suppose that a_3^k is a syllabus in β which is proceeded immediately by a_1. Then one can eliminate the syllabus a_3^k with the equality $a_1 a_3^k = a_2^k a_1$ to get an isotopic braid which is still in P^* but with one less number of syllabuses in a_3. Similarly if a syllabus a_3^k is followed immediately by a_2, then one can eliminate the syllabus a_3^k with the equality $a_3^k a_2 = a_2 a_1^k$ to get an isotopic braid which is still in P^* but with one less number of syllabuses in a_3. We shall call this process *index-3 reduction.* So for any given $\beta \in P^*$, we can find, after a finitely many times of index-3 reduction, an equivalent braid representative β' in P^* for β such that every syllabus in a_3 occurring in β' can only possibly be proceeded immediately by a_2 and likewise can only possibly be followed immediately by a_1. We call a word β in P^* *index-3 reduced* if every syllabus in a_3 occurring in β is neither proceeded immediately by a_1 nor followed immediately by a_2.

Let P^a denote the set of 3-braids of the form $\beta = a_i^{-q}\delta$, where $q = 0$ or 1 and $\delta \in P^*$ is a non-empty word, such that δ is index-3 reduced and β is cyclically syllabus reduced. Obviously P is contained in P^a.

Proposition 2. *Suppose that K is a knot in S^3 which is the closure of a 3-braid $\beta = a_i^{-q}\delta$ in P^a such that δ contains at least four syllabuses but β is not one of the words in the set*

$$\begin{aligned} E = \{&a_1^{-1}a_2a_3^2a_1a_2, a_1^{-1}a_3^2a_1a_2a_3, a_1^{-1}a_3a_1a_2^2a_3, a_1^{-1}a_2a_3a_1a_2^2,\\ &a_2^{-1}a_3a_1a_2a_3^2, a_2^{-1}a_3a_1^2a_2a_3, a_2^{-1}a_1a_2a_3^2a_1, a_2^{-1}a_1^2a_2a_3a_1,\\ &a_3^{-1}a_1a_2a_3a_1^2, a_3^{-1}a_1a_2^2a_3a_1, a_3^{-1}a_2a_3a_1^2a_2, a_3^{-1}a_2^2a_3a_1a_2\}\,. \end{aligned}$$

Then K has positive Casson invariant and thus has Property P.

Later on we shall refer the set E given in Proposition 2 as the *excluded* set.

Proposition 3. *Suppose that K is a knot in S^3 which is the closure of a 3-braid $\beta = \delta\eta$ which is in normal form* (iii)*, i.e. $\delta \in P$ and $\eta \in N$. Suppose that either*

1(1) *each of δ and η has a syllabus of power larger than one, or*
(2) *each of δ and η contains at least two syllabuses, or*
(3) *one of δ and η contains at least four syllabuses and the other has length at least two.*

Then each of $K(1)$ and $K(-1)$ is a manifold which contains an essential lamination.

If a closed 3-manifold has an essential lamination, then its universal cover is $\mathbb{R}^3$ [10] and thus in particular the manifold cannot be simply connected. Hence any knot as given in Proposition 3 has Property P by [5].

Given Propositions 2 and 3, we can finish the proof of Theorem 1 as follows. For a braid β, we use $\hat{\beta}$ to denote the closure of β. Let $K \subset S^3$ be a nontrivial knot with index 3. Let β be a 3-braid in normal form such that $\hat{\beta} = K$.

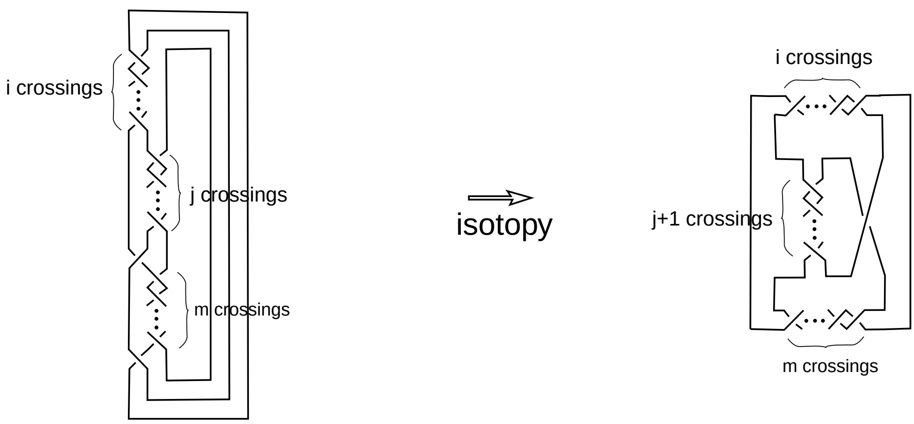


Fig. 2. The closure of $a_1^i a_2^j a_3^m$ is a Montesinos link.

First we consider the case that β is in normal form (i), i.e. $\beta = \alpha^k\delta$ with $\delta \in P$ and $k \geq 0$. If β contains at least four syllabuses and belongs to P^a, then $K = \hat{\beta}$ has positive Casson invariant by Proposition 2. So the knot K has Property P in this case. If β has less than four syllabuses, then up to conjugation in B_3, $\beta = a_2 a_1 a_3^i$ or $\beta = a_1^i a_2^j a_3^m$ for $i, j, m \geq 0$. It is easy to see that in this case $\hat{\beta}$ is an arborescent knot and thus by [17], $K = \hat{\beta}$ has Property P. For instance if $\beta = a_1^i a_2^j a_3^m$, then $\hat{\beta}$ is as shown in Fig. 2 which shows in fact that $\hat{\beta}$ is a Montesinos knot. So we may assume that β has at least four syllabuses but is not in P^a. This implies that in $\beta = \alpha^k \delta$, we have $k > 0$ and δ starts with a syllabus in a_3. Performing index-3 reduction on β once, we get an equivalent 3-braid $\beta' \in P^*$ which is index-3 reduced, i.e. $\beta' \in P^a$. If β' has less than four syllabuses, then again $K = \hat{\beta} = \hat{\beta}'$ is an arborescent knot and thus has Property P. If β' contains at least four syllabuses, we may apply Proposition 2 to get Property P for the knot.

If β is of normal form (ii), then the mirror image of β is a braid of normal form (i) and thus the knot $K = \hat{\beta}$ has Property P.

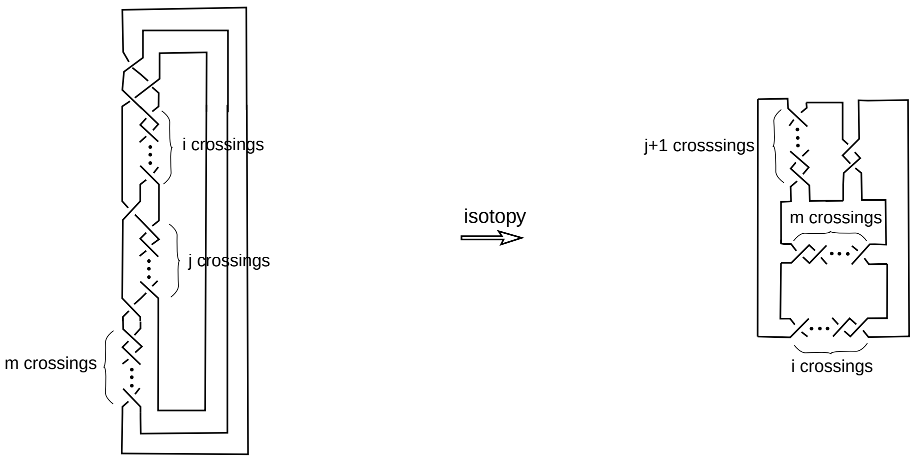


Fig. 3. The closure of $a_3^{-1} a_2^i a_3^j a_1^m$ is an arborescent link.

Suppose finally that $\beta = \delta\eta$ is a braid of normal form (iii) (recall that $\delta \in P$ and $\eta \in N$). By Proposition 3, we may assume that each of the conditions (1)-(3) in Proposition 3 does not hold for $\beta = \delta\eta$. Then β is a word of the form δa_i^{-1} or $a_i \eta$, or $\beta = \delta a_i^{-k}$ where $k > 1$ and δ contains at most three syllabuses each having power 1, or $\beta = a_1^k \eta$ where $k > 1$ and η contains at most three syllabuses each having power -1. If β contains at most four syllabuses, then $K = \hat{\beta}$ is an arborescent knot and thus has Property P. (Figure 3 shows this for the case that $\beta = a_3^{-1} a_2^i a_3^j a_1^m$. Other cases can be treated similarly). So we may only consider the cases when $\beta = \delta a_i^{-1}$ or $\beta = a_i \eta$, each containing at least five syllabuses. If $\beta = \delta a_i^{-1}$, then β is conjugate to $\beta' = a_i^{-1} \delta$ which is in P^a. Hence if β' does not belong to the excluded set

$$\begin{aligned} E = \{ & a_1^{-1} a_2 a_3^2 a_1 a_2, a_1^{-1} a_3^2 a_1 a_2 a_3, a_1^{-1} a_3 a_1 a_2^2 a_3, a_1^{-1} a_2 a_3 a_1 a_2^2, \\ & a_2^{-1} a_3 a_1 a_2 a_3^2, a_2^{-1} a_3 a_1^2 a_2 a_3, a_2^{-1} a_1 a_2 a_3^2 a_1, a_2^{-1} a_1^2 a_2 a_3 a_1, \\ & a_3^{-1} a_1 a_2 a_3 a_1^2, a_3^{-1} a_1 a_2^2 a_3 a_1, a_3^{-1} a_2 a_3 a_1^2 a_2, a_3^{-1} a_2^2 a_3 a_1 a_2 \}, \end{aligned}$$

then $K = \hat{\beta}'$ has positive Casson invariant by Proposition 2 and thus has Property P. If β' is in the excluded set E, then $K = \hat{\beta}'$ is an arborescent knot and thus has

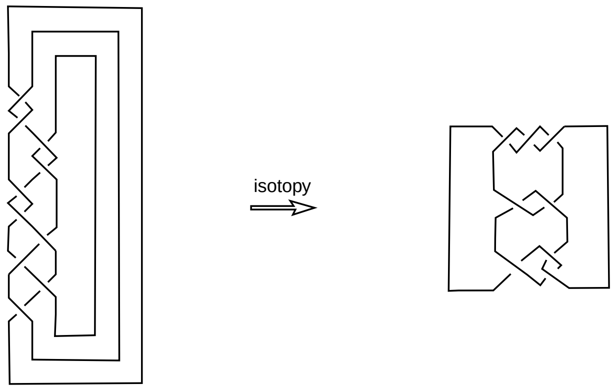


Fig. 4. The closure of the braid $a_1^{-1}a_3^2a_1a_2a_3$ is a Montesinos knot.

Property P [17]. (Figure 4 illustrates this for the case $\beta = a_1^{-1}a_3^2a_1a_2a_3$. Other cases can be checked similarly). Finally, if $\beta = a_i\eta$, then its mirror image is a braid in P^a, which is a case we have just discussed. This completes the proof of Theorem 1.

Propositions 2 and 3 will be proved in subsequent two sections which constitutes the rest of the paper.

3. Proof of Proposition 2

We retain all definitions and notations established earlier. For a 3-braid β, we use n_β to denote the number of syllabuses in a_3 occurring in β and use s_β to denote the number of syllabuses of β. Obviously if β' is the braid obtained from $\beta \in P^*$ after some non-trivial index-3 reduction, then $\hat{\beta}' = \hat{\beta}$ but $n_{\beta'} < n_\beta$. For a syllabus $a_3^k = a_1^{-1}a_2^ka_1$, we shall always assume its plane projection corresponds naturally to $a_1^{-1}a_2^ka_1$. Hence, every 3-braid in letters a_1, a_2, a_3 has its *canonical* plane projection; namely in the projection plane we place vertically (from top to bottom) and successively the projections of letters occurring in the braid, corresponding to their natural order from left to right. Whenever we need consider a plane projection of a 3-braid, the canonical one is always assumed unless specifically indicated otherwise.

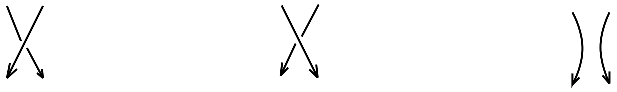


Fig. 5. K_+ (the left figure), K_- (the middle figure) and L_0 (the right figure).

The basic tool we are going to use to prove the proposition is the following crossing change formula $(*)$ of the Casson invariant. If K_+, K_- are oriented knots and L_0 an oriented link with two components in S^3 such that they have identical plane projection except at one crossing they differ as shown in Fig. 5, then the

Casson invariants C_{K_+} and C_{K_-} of K_+ and K_- satisfy the following relation:

$$C_{K_+} - C_{K_-} = lk(L_0)\,, \qquad (*)$$

where $lk(L_0)$ is the linking number of L_0. This formula can be found on page 141 of [1] (note that there was a print error there, K_+ and K_- in Figs. 36 and 37 of [1] should be exchanged). The idea of proof of the proposition is repeatedly applying the formula $(*)$ to a 3-braid of the type as given in the proposition to reduce the complexity of the braid and inductively prove the positivity of its Casson invariant. To do so, we first need to estimate the linking number of a two-component link which is the closure of a 3-braid of relevant type. Given an oriented link L of two components L_1 and L_2, we shall calculate the linking number of L as follows. Take a plane projection of L. A crossing of L as shown on the left of Fig. 5 has positive sign 1 and the crossing in the middle of the figure has negative sign -1. Let p be the number of positive crossings between of L_1 and L_2 and q the number of negative crossings between L_1 and L_2. The linking number of L is equal to $(p-q)/2$. In other words, each positive (respectively negative) crossing between L_1 and L_2 contributes $1/2$ (respectively $-1/2$) to the linking number of L, and any crossing between L_1 and L_1 or between L_2 and L_2 contributes 0 to the linking number of L.

For a 3-braid β, we shall always orient each component of $\hat{\beta}$ in such a way that the induced orientation on each strand of β in its canonical projection is pointing downward in the projection plane. Suppose that $\hat{\beta}$ is a link of two components. If β' is a portion of β, we use $l(\beta')$ to denote the total contribution to the linking number of $\hat{\beta}$ coming from all the crossings of β', which maybe a half integer. In particular $l(\beta) = lk(\hat{\beta})$. Also if we decompose β into portions $\beta = \beta_1\beta_2\cdots\beta_k$, then $lk(\hat{\beta}) = l(\beta_1) + l(\beta_2) + \cdots + l(\beta_k)$.

Given the canonical plane projection of a 3-braid η, we shall always call the strand of η which starts at the top left corner *the strand 1* of η, call the strand which starts at the top middle place *the strand 2* of η, and call the strand which starts at the top right corner *the strand 3* of η.

Lemma 4. *Let $L = L_1 \cup L_2$ be a link of two components in S^3 which is the closure of a braid $\beta = a_i^{-q}\delta$, where $q = 0$ or 1, and δ is a positive word in a_1, a_2, a_3. Then the linking number $lk(L)$ of L is non-positive.*

Proof. Note that $l(a_i) \le 0$ for $i = 1, 2, 3$, and $l(a_i^{-1}) \le 1/2$ for $i = 1, 2$. Hence if in $\beta = a_i^{-q}\delta$, $q = 0$ or $q = 1$ but $i = 1$ or 2, then $lk(L) = l(\beta) = l(a_i^{-q}) + l(\delta) \le 1/2$. But $lk(L)$ is an integer, so $lk(L) \le 0$. So we may assume that $\beta = a_3^{-1}\delta$ and also assume that $l(a_3^{-1}) = 1$. So in turn we may assume that the strand 1 of a_3^{-1} belongs to L_1 and both strand 2 and 3 of a_3^{-1} belong to L_2. One can then easily check that for any positive word δ in a_1, a_2, a_3, that one must have $l(\delta) < 0$, in order to have a consistent link component assignment to the strands of β (one can see this by just looking at the first one or two possible syllabuses of δ). Hence $lk(L) = l(\beta) \ge 0$. $\square$

We now prove Proposition 2. Let $\beta = a_i^{-q}\delta$ be a 3-braid in P^a as given in the proposition, whose closure is the given knot K. Recall that $s_\delta \geq 4$. Let l_δ denote the total length of δ. The stratege of the proof is to use induction on the total length l_δ of a knot from the set all braids $\beta = a^{-1}\delta$ in P^a with $s_\delta \geq 4$, applying the crossing change formula for Casson invariant.

By hand calculation case by case, one can check that Proposition 2 holds for all braids $\beta = a_i^{-q}\delta \in P^a$ with $s_\delta \geq 4$ and $l_\delta \geq 5$. So we may assume that a given $\beta = a_i^{-1}\delta \in P^a$ has $s_\delta \geq 4$ and $l_\delta \geq 6$.

If some syllabus in a_3 occurring in δ has power $k > 2$, we apply the crossing change formula $(*)$ for Casson invariant to $K = \hat{\beta}$ at the second crossing of the syllabus. The link L (of two components) obtained by smoothing the crossing is the closure of a braid of the type as described in Lemma 4 and thus has non-positive linking number. So we get a new braid $\beta' = a_i^{-q}\delta'$ which is identical with β except with two less in power at the syllabus in a_3 and the Casson invariant of $\hat{\beta}'$ is less than or equal to that of $\hat{\beta}$. Obviously β' is still in P^a. So if $l_\delta \geq 6$, we may apply induction. If $l_{\delta'} < 6$ we may check directly that the original braid $\hat{b}$ has positive Casson invariant. So we may assume that every syllabus in a_3 has power at most two. Similarly we may assume that the power of a syllabus in a_2 or in a_1 is one or two.

Claim D1. *If a_3^2 is a syllabus of δ, then the Proposition holds.*

Proof. Consider the first such syllabus occurring in δ. Applying the formula $(*)$ to $\hat{\beta}$ at the second crossing of the syllabus a_3^2, we get a new braid $\beta' = a_i^{-q}\delta'$, where δ' is the braid obtained from δ by deleting the syllabus a_3^2, such that $\delta' \in P^*$, $n_{\delta'} = n_\delta - 1$, and $C_{\hat{\beta}} \geq C_{\hat{\beta}'}$ (by Lemma 4). If β' is still in P^a and δ' contains at least four syllabuses but β' is not a word in the excluded set E, then we may use induction.

Suppose that $\beta' \in E$. Since the syllabus we are considering is the first such occurring in β and since δ is index-3 reduced, β can only be the word $a_2^{-1}a_3^2a_1a_2a_3^2a_1$ or $a_2^{-1}a_3^2a_1^2a_2a_3a_1$ or $a_2^{-1}a_3^2a_1a_2a_3a_1^2$ or $a_1^{-1}a_2a_3a_1a_2^2a_3^2$. In such a case one can verify directly that $C_{\hat{\beta}}$ is positive.

Hence, we may assume that either β' is still in P^a with δ' containing exactly three syllabuses, or β' is no longer in P^a. In the former case, β is a word in the set

$$\left\{\begin{array}{lllll} a_1^ja_2^ka_3^2a_1^m, & a_2^ja_1^ka_2^ma_3^2, & a_2^ja_3^2a_1^ka_2^m, & a_3^2a_1^ka_2^ja_3^m, & a_3^2a_1^ka_2^ja_1^m, \\ a_3a_1^ka_2^ma_3^2, & a_1^{-1}a_2^ja_1^ka_2^ma_3^2, & a_1^{-1}a_2^ja_3^2a_1^ka_2^m, & a_1^{-1}a_3a_1^ka_2^ma_3^2, & a_2^{-1}a_1^ja_2^ka_3^2a_1^m, \\ a_2^{-1}a_3^2a_1^ja_2^ka_3^m, & a_2^{-1}a_3^2a_1^ka_2^ja_1^m, & a_3^{-1}a_1^ja_2^ka_3^2a_1^m, & a_3^{-1}a_2^ja_3^2a_1^ka_2^m & \end{array}\right\}$$

for some $j, k, m \in \{1, 2\}$. When β is one of words in this set but is not in the excluded set E, one can verify directly using the formula $(*)$ that $\hat{\beta}$ has positive Casson invariant. For instance, when $\beta = a_1^{-1}a_2^ja_1^ka_2^ma_3^2$, it is conjugate to $a_2^ja_1^ka_2^ma_3^2a_1^{-1} = a_2^ja_1^ka_2^ma_1^{-1}a_2^2$ which in turn is conjugate to $a_1^{-1}a_2^{j+2}a_1^ka_2^m$. So we need to show that the Casson invariant of the closure of the braid $\eta = a_1^{-1}a_2^{j+2}a_1^ka_2^m$ is positive. We have eight possible cases for η corresponding to various possible values of j, k and m. But only in case $j = k = m = 1$ or case $j = 1, k = m = 2$ or case $j = k = 2, m = 1$, the closure of the involved braid is a knot, and in such a case one can verify directly using formula $(*)$ that $\hat{\beta} = \hat{\eta}$ has positive Casson invariant. In a similar way one can verify the proposition for each of the other words in the above set. (We did the checking!)

We now consider the latter case when β' is not in P^a. It follows that the syllabus a_3^2 is either the first or the last syllabus of δ and $q = 1$. We consider the case when the syllabus a_3^2 is the first syllabus of δ. The case when the syllabus a_3^2 is the last syllabus of δ can be treated similarly. It follows that $\beta = a_1^{-1}a_3^2a_1^ja_2^k\cdots$, for some $j, k \in \{1, 2\}$, and β does not end with a_1. If $j = 2$, then $\beta' = a_1^{-1}a_1^2a_2^k\cdots$ which is isotopic to $\beta'' = a_1a_2^k\cdots$ which is in P^a. So if β'' contains at least four syllabuses, we may use the induction or check $C_{\hat{\beta}}$ directly. We may then assume that β'' has less than four syllabuses. It follows that $\beta'' = a_1a_2^ka_3^m$ which is a knot only when $k = 2, m = 1$ or $k = 1, m = 2$ and in these two cases $C_{\hat{\beta}} \geq C_{\hat{\beta}''} > 0$. So we may assume that $j = 1$. In this case β' is isotopic to $\beta'' = a_2^k\cdots$ which is in P^a. Hence if β'' contains at least four syllabuses, we may use the induction. If β'' has less than four syllabuses, then we must either have $\beta = a_1^{-1}a_3^2a_1a_2^ka_1^na_2^m$ or $\beta = a_1^{-1}a_3^2a_1a_2^ma_3^j$. In the former case, β'' has positive Casson invariant (a nontrivial positive knot). In the latter case, only when $m = 1, j = 1$, $\hat{\beta}$ is a knot. But this braid is in the set E. The proof of Claim D1 is now complete. □

By Claim D1, we may now assume that every syllabus in a_3 occurring in δ has power equal to one.

Claim D2. *We may assume that every syllabus in a_1 occurring in δ has power equal to one.*

Proof. Suppose that δ contains syllabuses in a_1 of power two. Consider the first such syllabus occurring in δ. Applying the formula $(*)$ to $\hat{\beta}$ at the first crossing of the syllabus a_1^2, we get a new braid $\beta' = a_i^{-q}\delta'$ such that the Casson invariant of $\hat{\beta}'$ is less than or equal to that of $\hat{\beta}$ by Lemma 4. Obviously δ' is still a positive word in a_1, a_2, a_3 but may not be in P^* or P^a. We have several possibilities for δ around the given syllabus a_1^2: $\delta = \cdots a_2^ja_1^2a_2^k\cdots$ or $\delta = \cdots a_3a_1^2a_2^k\cdots$ or $\delta = a_1^2a_2^k\cdots$ or $\delta = \cdots a_1^2$, for some $j, k \in \{1, 2\}$.

Case (D2.1). $\delta = \cdots a_2^ja_1^2a_2^k\cdots$.

Then $\delta' = \cdots a_2^{j+k}\cdots$ and $\beta' = a_i^{-q}\delta'$ is still in P^a. Also if $j + k > 2$, we may apply the formula $(*)$ one more time to bring it down to one or two. Let $\beta'' = a_i^{-q}\delta''$ be the resulting braid. Then if δ'' contains at least four syllabuses, we may use the induction or check $C_{\hat{\beta}}$ directly. If β'' is in E, then one can verify directly that the original braid β has positive Casson invariant. Note that $s_{\delta'} = s_\delta - 2$. Suppose that $s_{\delta'} < 4$. Then s_δ is four or five and $\beta = a_i^{-q}a_2^ja_1^2a_2^ka_3$ or $\beta = a_i^{-q}a_1a_2^ja_1^2a_2^ka_3$ or

$\beta = a_i^{-q} a_3 a_1 a_2^j a_1^2 a_2^k$. Easy to check that when $q = 0$, any knot from these cases has positive Casson invariant. So assume that $q = 1$ in these cases.

If $\beta = a_i^{-1} a_2^j a_1^2 a_2^k a_3$, then $i = 1$ since β is cyclically reduced. So $\beta = a_1^{-1} a_2^j a_1^2 a_2^k a_3$. To be a knot, we have $k = j = 1$ or $k = j = 2$. In each of the two cases, one can verify directly that the Casson invariant of the knot is positive.

If $\beta = a_i^{-q} a_1 a_2^j a_1^2 a_2^k a_3$, then $i = 2$. That is $\beta = a_2^{-1} a_1 a_2^j a_1^2 a_2^k a_3$. One can also directly verify that $C_{\hat\beta} > 0$ (for those values of $k, j \in \{1, 2\}$ which make $\hat\beta$ a knot).

The case that $\beta = a_i^{-q} a_3 a_1 a_2^j a_1^2 a_2^k$ can be treated similarly.

Case (D2.2). $\delta = \cdots a_3 a_1^2 a_2^k \cdots$.

Then $\delta' = \cdots a_3 a_2^k \cdots = \cdots a_2 a_1 a_2^{k-1} \cdots$.

Case (D2.2.1.) $k = 2$.

If δ does not start with a_3, then $\beta' = a_i^{-q}\delta' = a_i^{-q} \cdots a_2^{j+1} a_1 a_2 \cdots$ is still in P^a and is not in the set E. In such case if δ' contains at least four syllabuses, we may apply induction or check $C_{\hat\beta}$ directly. If $s_{\delta'} < 4$, then $\beta = a_i^{-q} a_2^j a_3 a_1^2 a_2^2$. To be a knot, we have $q = 0, j = 1$ or $q = 1, i = 1, j = 2$. In each of the two cases we have $C_{\hat\beta} > 0$ by direct calculation.

If δ starts with a_3 but $q = 0$, then $\beta' = \delta'$ is still in P^a but not in E. Also $s_\beta = s_\delta = s_{\beta'} = s_{\delta'}$. So we may apply induction or check $C_{\hat\beta}$ directly.

If δ starts with a_3 and $q = 1$, then $i = 1$ or 2. If $i = 1$, then $\beta' = a_1^{-1} a_2 a_1 a_2 \cdots$ is still in P^a but not in E and contains at least five syllabuses. So we may use induction. If $i = 2$, then $\beta' = a_1 a_2 \cdots$ is in P^a and is not in E. Also $s_{\beta'} = s_\delta - 1$. Hence if $s_\delta > 4$, we may use induction. So suppose that $s_\delta = 4$. Then $\beta = a_2^{-1} a_3 a_1^2 a_2^2 a_3$ or $\beta = a_2^{-1} a_3 a_1^2 a_2^2 a_1^j$. But the former is not a knot. The latter is a knot when $j = 2$, in which case $C_{\hat\beta} > 0$.

Case (D2.2.2). $k = 1$.

Then $\delta = \cdots a_3 a_1^2 a_2 a_3 \cdots$ or $\delta = \cdots a_3 a_1^2 a_2 a_1^j \cdots$ or $\delta = \cdots a_3 a_1^2 a_2$. Correspondingly, we have $\delta' = \cdots a_2 a_1 a_3 \cdots = \cdots a_2^2 a_1 \cdots$ or $\delta = \cdots a_2 a_1^{j+1} \cdots$ or $\delta = \cdots a_2 a_1$. If β' is in $P^a - E$ and $s_{\delta'} \geq 4$, we may use induction. If $\beta' \in E$, then one can check that the old braid β has positive Casson invariant. If $\beta' \in P^a$ but $s_{\delta'} < 4$, then one can also verify that β always has positive Casson invariant, applying the conditions that (1) $\beta \in P^a - E$, (2) $s_\delta \geq 4$ and (3) $\hat\beta$ is a knot.

Case (D2.3). $\delta = a_1^2 a_2^k \cdots$.

Then $\delta = a_1^2 a_2^k a_1^j a_2^m \cdots$ or $\delta = a_1^2 a_2^k a_3 a_1^j \cdots$. And $\delta' = a_2^k a_1^j a_2^m \cdots$ or $\delta' = a_2^k a_3 a_1^j \cdots$. So β' is in P^a unless β starts with a_2^{-1}. If β starts with a_2^{-1}, then $\beta' = a_2^{k-1} a_1^j a_2^m \cdots$ or $\beta' = a_2^{k-1} a_3 a_1^j \cdots$ which is in P^a. Again in each of these cases, if β' is the set E or $s_{\delta'} < 4$, one can calculate directly that the old braid β has positive Casson invariant. Otherwise one can use the induction.

Case (D2.4). $\delta = \cdots a_1^2$.

This case can be treated similarly as in the previous case. □

So by Claims D1 and D2, we now assume that every syllabus in a_3 and in a_1 occurring in δ have power equal to one. $\square$

Claim D3. *We may assume that every syllabus in a_2 occurring in δ has power equal to one.*

Proof. This can be proved similarly as Claim D2. $\square$

So we now assume that every syllabus occurring in δ has power one.

If $a_1a_2a_1$ appears immediately after an a_3, then $\beta = \cdots a_3a_1a_2a_1 \cdots = \cdots a_3a_1a_3a_2 \cdots = \cdots a_3a_2a_1a_2 \cdots = \cdots a_2a_1^2a_2 \cdots$ which is still in P^a. Hence we may apply Claim D2 β is in the set $\{a_3a_1a_2a_1,\ a_2^{-1}a_3a_1a_2a_1,\ a_3^{-1}a_2a_3a_1a_2a_1,\ a_1^{-1}a_3a_1a_2a_1a_2,\ a_1^{-1}a_2a_3a_1a_2a_1a_2,\ a_3^{-1}a_2a_3a_1a_2a_1a_2\}$. If β is a word in this set, then either $\hat{\beta}$ is not a knot or $\hat{\beta}$ has positive Casson invariant. So we may assume that no $a_3a_1a_2a_1$ occurs in β. A similar argument shows that we may assume that no $a_2a_1a_2a_3$ occurs in β. Hence we may assume that β is one of the words in the set

$$\left\{\begin{array}{llll} a_1a_2a_3a_1, & (a_3a_1a_2)^m, & (a_3a_1a_2)^m a_3, & (a_3a_1a_2)^m a_3a_1, \\ a_2(a_3a_1a_2)^m, & a_1a_2(a_3a_1a_2)^m, & a_2(a_3a_1a_2)^m a_3, & a_2(a_3a_1a_2)^m a_3a_1, \\ a_1a_2(a_3a_1a_2)^m a_3, & a_1a_2(a_3a_1a_2)^m a_3a_1, & a_2^{-1}a_1a_2a_3a_1, & a_3^{-1}a_1a_2a_3a_1, \\ a_1^{-1}(a_3a_1a_2)^m, & a_1^{-1}(a_3a_1a_2)^m a_3, & a_2^{-1}(a_3a_1a_2)^m a_3, & a_2^{-1}(a_3a_1a_2)^m a_3a_1 \\ a_1^{-1}a_2(a_3a_1a_2)^m, & a_3^{-1}a_2(a_3a_1a_2)^m, & a_3^{-1}a_1a_2(a_3a_1a_2)^m, & a_1^{-1}a_2(a_3a_1a_2)^m a_3, \\ a_3^{-1}a_2(a_3a_1a_2)^m a_3a_1 & a_2^{-1}a_1a_2(a_3a_1a_2)^m a_3, & a_2^{-1}a_1a_2(a_3a_1a_2)^m a_3a_1, & a_3^{-1}a_1a_2(a_3a_1a_2)^m a_3a_1 \end{array}\right\}$$

where $m > 0$.

If $\beta = a_1a_2a_3a_1$, then it has positive Casson invariant. The case $\beta = (a_3a_1a_2)^m$ cannot occur since its closure is not a knot for all $m > 0$. Similarly each of the cases

$$a_2(a_3a_1a_2)^m a_3a_1,\ a_1a_2(a_3a_1a_2)^m a_3,\ a_2^{-1}a_1a_2a_3a_1,$$

$$a_3^{-1}a_1a_2a_3a_1,\ a_1^{-1}(a_3a_1a_2)^m,\ a_2^{-1}(a_3a_1a_2)^m a_3a_1 \quad \text{and} \quad a_1^{-1}a_2(a_3a_1a_2)^m a_3$$

cannot occur as β. If $\beta = (a_3a_1a_2)^m a_3$, then it is conjugate to $\beta' = a_3^2a_1a_2(a_3a_1a_2)^{m-1}$. So we may apply Claim D1 unless $m = 1$. But when $m = 1$, one can calculate directly that $\beta = a_3a_1a_2a_3$ has positive Casson invariant. If $\beta = (a_3a_1a_2)^m a_3a_1$, then it is conjugate to $\beta' = a_2a_1^3a_2(a_3a_1a_2)^{m-1}$. So we are back to a previous case unless $m = 1$. But when $m = 1$, $\hat{\beta}$ is not a knot. If $\beta = a_2(a_3a_1a_2)^m$, then it is conjugate to $\beta' = (a_3a_1a_2)^{m-1}a_3a_1a_2^2$. So we may apply Claim D2 unless $m = 1$. But when $m = 1$, one can calculate directly that $\beta = a_2a_3a_1a_2$ has positive Casson invariant. Similarly one can deal with the cases $a_1a_2(a_3a_1a_2)^m$, $a_2(a_3a_1a_2)^m a_3$ and $a_1a_2(a_3a_1a_2)^m a_3a_1$.

Suppose that $\beta = a_1^{-1}(a_3a_1a_2)^m a_3$. To be a knot, we have $m \geq 3$ and $m = 3$ (mod 2). Also β is conjugate to $\beta' = a_1^{-1}a_2(a_3a_1a_2)^m$ which has less number of syllabuses in a_3 and thus we may apply the induction. Similarly one can deal with the case when $\beta = a_2^{-1}(a_3a_1a_2)^m a_3$.

If $\beta = a_1^{-1}a_2(a_3a_1a_2)^m$, then to be a knot m must be even. Applying the formula $(*)$ at the first crossing of β, we get $C_{\hat{\beta}} = C_{\hat{\beta}_1} + lk(\hat{\lambda}_1)$ where $\beta_1 = a_1a_2(a_3a_1a_2)^m$

and $\lambda_1 = a_2(a_3a_1a_2)^m$. One can easily deduce from the projection of λ_1 that $lk(\hat{\lambda}_1) = -m/2$. Hence, we have $C_{\hat{\beta}} = C_{\hat{\beta}_1} - m/2$. In such case it suffices to show the following.

Claim D4. $C_{\hat{\beta}_1} > m/2$.

Proof. We knew that $m = 2p$ with $p > 0$. We shall prove the claim by induction on the number p. Write β_1 as $\beta_1 = a_1a_2a_1^{-1}a_2a_1^2a_2a_1^{-1}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$. Applying formula $(*)$ to β_1 at the first crossing of the first syllabus a_1^2, we get $C_{\hat{\beta}_1} = C_{\hat{\beta}_2} - lk(\hat{\lambda}_2)$ where $\beta_2 = a_1a_2a_1^{-1}a_2^2a_1^{-1}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$ and $\lambda_2 = a_1a_2a_1^{-1}a_2a_1a_2a_1^{-1}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$. One can easily deduce from the projection of λ_2 that $lk(\hat{\lambda}_2) = -2(m-2) - 3 = -4p + 1$. Hence, we have $C_{\hat{\beta}_1} = C_{\hat{\beta}_2} + 4p - 1$. We then apply formula $(*)$ to β_2 at the first crossing of the first syllabus a_2^2, we get $C_{\hat{\beta}_2} = C_{\hat{\beta}_3} - lk(\hat{\lambda}_3)$ where $\beta_3 = a_1a_2a_1^{-2}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$ and $\lambda_3 = a_1a_2a_1^{-1}a_2a_1^{-1}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$. One can calculate to see that $lk(\hat{\lambda}_3) = -p - 1$. Hence, we have $C_{\hat{\beta}_2} = C_{\hat{\beta}_3} + p + 1$. We then apply formula $(*)$ to β_3 at the first crossing of the syllabus a_1^{-2}, we get $C_{\hat{\beta}_3} = C_{\hat{\beta}_4} + lk(\hat{\lambda}_4)$ where $\beta_4 = a_1a_2^2a_1^2a_2(a_3a_1a_2)^{m-2}$ and $\lambda_4 = a_1a_2a_1^{-1}a_2a_1^2a_2(a_3a_1a_2)^{m-2}$. Also one can calculate to see that $lk(\hat{\lambda}_4) = -p$. Hence, we have $C_{\hat{\beta}_3} = C_{\hat{\beta}_4} - p$. We then apply formula $(*)$ to β_4 at the first crossing of the syllabus a_2^2, we get $C_{\hat{\beta}_4} = C_{\hat{\beta}_5} - lk(\hat{\lambda}_5)$ where $\beta_5 = a_1^3a_2(a_3a_1a_2)^{m-2}$ and $\lambda_5 = a_1a_2a_1^2a_2(a_3a_1a_2)^{m-2}$. We also have $lk(\hat{\lambda}_5) = -4(p-1) - 2$. Hence we have $C_{\hat{\beta}_4} = C_{\hat{\beta}_5} + 4(p-1) + 2$. Applying formula $(*)$ to β_5 at the first crossing of the syllabus a_1^2, we get $C_{\hat{\beta}_5} = C_{\hat{\beta}_6} - lk(\hat{\lambda}_6)$ where $\beta_6 = a_1a_2(a_3a_1a_2)^{m-2}$ and $\lambda_6 = a_1^2a_2(a_3a_1a_2)^{m-2}$. We also have $lk(\hat{\lambda}_6) = -(p-1) - 1 = -p$. Hence we have $C_{\hat{\beta}_5} = C_{\hat{\beta}_6} + p$. In summary, we get

$$C_{\hat{\beta}_1} = C_{\hat{\beta}_6} + 9p - 2.$$

Now one can easily see that the claim follows. □

Similarly we can treat the rest of cases. The proof of Proposition 2 is now complete.

4. Proof of Proposition 3

Given a 3-braid β in letters a_1, a_2, a_3, whose closure is a knot, there is a canonical way to construct a Seifert surface for $\hat{\beta}$ as follows: in the projection plane we have the braid diagram in its canonical form, place three rectangular disks in the space so that disk 1 lies in the projection plane and is on the left hand side of the braid, disk 3 also lies in the projection plane but on the right hand side of the braid, disk 2 lies perpendicularly above the projection plane, each disk having one side running parallel to the braid from top to the bottom, then to each letter a_1 (a_1^{-1}) occurring in β use a half negatively (positively) twisted band connecting disks 1 and 2, to each letter a_2 (a_2^{-1}) use a half negatively (positively) twisted band connecting disks

2 and 3, and to each letter a_3 (a_3^{-1}) use a half negatively (positively) twisted band connecting disks 1 and 3 (behind disk 2). Figure 6 illustrates such construction for $\beta = a_1 a_2 a_3 a_1^{-1} a_3^{-1} a_2^{-1}$. We call the Seifert surface of $\hat{\beta}$ so constructed *canonical* Seifert surface of $\hat{\beta}$. In [18], it was proved that if β is a 3-braid in norm form (whose definition we recalled in Section 2), then its canonical Seifert surface has the minimal genus (thus is an essential and Thurston norm minimizing surface in the exterior of $\hat{\beta}$).

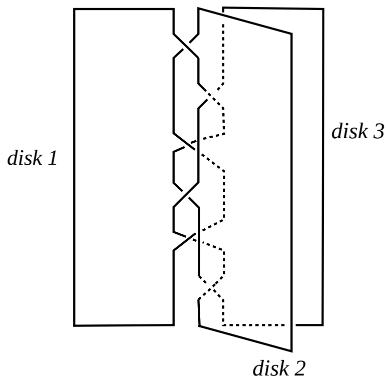


Fig. 6. Construction of the canonical Seifert surface.

Lemma 5. *Let β be a 3-braid such that $\hat{\beta}$ is a knot. Let S be the canonical Seifert surface of β and suppose that it has minimal genus.*

(1) *If S contains two half twisted bands corresponding to the same letter a_1 or a_2 or a_3, then the -1-surgery on $\hat{\beta}$ is a manifold with essential lamination.*
(2) *If S contains two half twisted bands corresponding to the same letter a_1^{-1} or a_2^{-1} or a_3^{-1}, then the 1-surgery on $\hat{\beta}$ is a manifold with essential lamination.*

Proof. We shall only prove part (1) when S contains two half twisted bands corresponding to the same letter a_1. All other cases can be proved similarly.

Let M be the knot exterior of $\hat{\beta}$ in S^3. We shall also use $M(-1)$ to denote the manifold obtained by Dehn surgery on the knot $\hat{\beta}$ with the slope -1. Let V be the solid torus filled in M to obtain the manifold $M(-1)$. We first construct an essential branched surface B in the exterior M and then prove that B (which has boundary on ∂M) can be capped off by a branched surface in V to yield an essential branched surface $\hat{B}$ in $M(-1)$. The construction of B is similar to that given in [14].

Since S contains two half negatively twisted bands corresponding to a_1, there is a disk D in M as shown in Fig. 7(1) whose boundary lies in $S \cup \partial M$ and whose interior is disjoint from $S \cup \partial M$. With more detail, the boundary of ∂D intersects S in two disjoint arcs and intersects ∂M in two disjoint arcs with the latter happening around

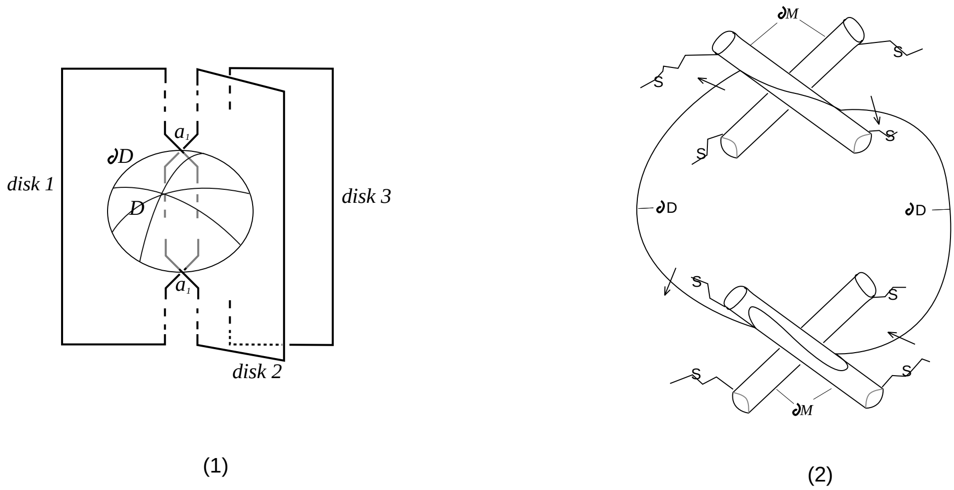


Fig. 7. Construction of B.

the places corresponding to the two bands of a_1. (Similar disks were used in [3] for a different purpose). The branched surface B is the union of the Seifert surface S and the disk D with their intersection locus smoothed as shown in Fig. 7(2). The arrows in the figure indicate the cusp direction of branched locus. A similar argument as in [14] shows that the branched surface fully carries a lamination with no compact leaves and each negative slope can be realized as the boundary slope of a lamination fully carried by B. Since the disk D intersects the knot exactly twice, the branched surface B is essential in M by [9, 3.12]. The branched locus of B is a set of two disjoint arcs properly embedded in S, each being non-separating. The branched surface B meets ∂M yielding a train track in ∂M as show in Fig. 8(1). Let $\mathcal{L}$ be a lamination fully carried by B whose boundary slope is -1. Then $\partial \mathcal{L}$ must look like that shown in Fig. 8(2).

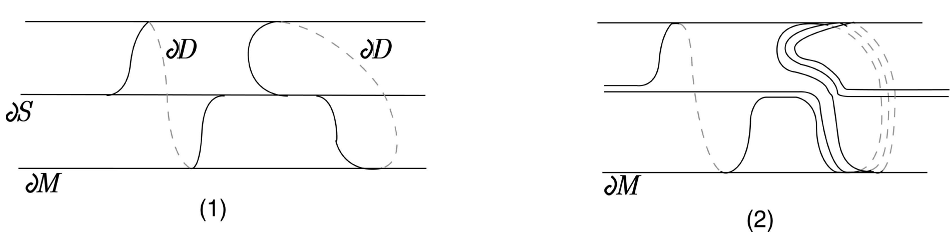


Fig. 8. (1) ∂B on ∂M, (2) the curve of slope -1 fully carried by ∂B.

We now construct a branched surface B_V in the sewn solid torus V such that the train track $B_V \cap \partial V$ is equivalent to the train track $B \cap \partial M$ and B_V fully carries a lamination which is a set of meridian disks of V. Hence, B and B_V match together and form a branched surface $\hat{B}$ in $M(-1)$. Take a meridian disk D_0 of V and push part of it near and around ∂V as shown in Fig. 9(1) and then identify two disjoint sub-disks of D_0 as shown in Fig. 9(2). This gives a branched surface B_1 with the cusp direction along its singular locus (an arc) as shown in Fig. 9(2). Then we split B_1 locally at a place around a point of ∂B_1 as shown in Fig. 9(3) and then we start pinch the resulting branched surface along part of its boundary as shown in Fig. 9(4). The pinching continues as shown in Fig. 9(5) until we get the branch surface whose boundary is as shown in Fig. 9(6). The resulting surface is the branched surface B_v. Obviously the train track ∂B_v on ∂V is equivalent to the train track ∂B in ∂M and they can be matched in $\partial V = \partial M$.

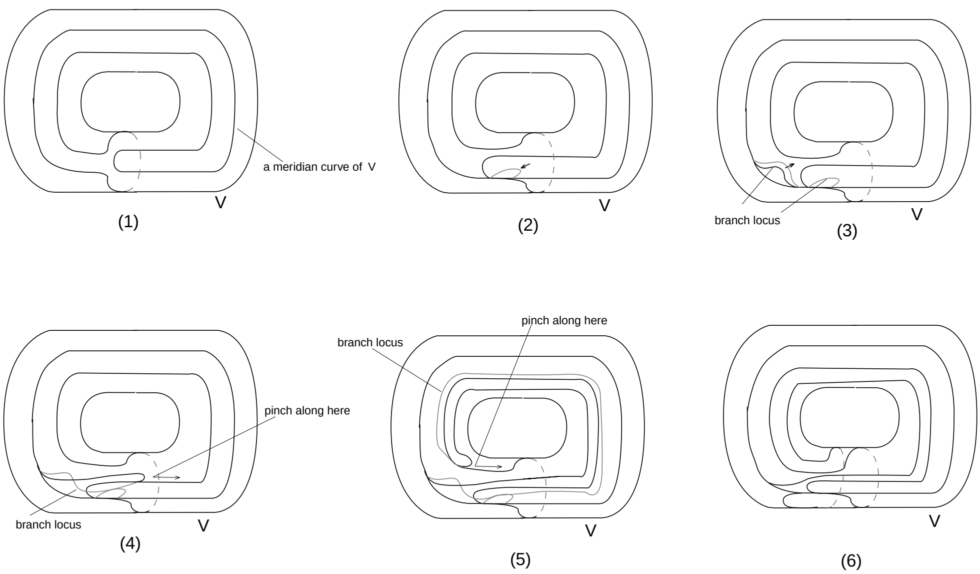


Fig. 9. Construction of B_V.

To see that $\hat{B}$ is essential we have five things to check by [10, Definition 2.1]:

(i) $\hat{B}$ has no discs of contact;

(ii) the horizontal surface $\partial_h N(\hat{B})$ is incompressible and ∂-incompressible in $M(-1) - \overset{\circ}{N}(\hat{B})$, there are no monogons in $M(-1) - \overset{\circ}{N}(\hat{B})$ and no component of $\partial_h N(\hat{B})$ is a 2-sphere;

(iii) $M(-1) - \overset{\circ}{N}(\hat{B})$ is irreducible;

(iv) $\hat{B}$ contains no Reeb branched surface;

(v) $\hat{B}$ fully carries a lamination.

Condition (v) follows automatically from the construction since leaves of a lamination fully carried by B with boundary slope -1 match on $\partial M = \partial V$ with (disk) leaves of a lamination fully carried by B_V. It also follows that $\hat{B}$ does not carry any compact surface since B does not. Hence in particular condition (iv) holds also for $\hat{B}$. By the construction, one can easily see that $V - \overset{\circ}{N}(B_v)$ has two components, each of which topologically looks like as shown in Fig. 10. It follows that $M(-1) - \overset{\circ}{N}(\hat{B})$ is topologically the same as $M - \overset{\circ}{N}(B)$, with the same horizontal surface. From this we get conditions (ii) and (iii) for $\hat{B}$.

We now show that $\hat{B}$ has no disk of contact. Note that $\partial_v(N(\hat{B}))$ is a set of two annuli and each of the annuli is obtained from matching a component of $\partial_v(N(B))$ (a vertical disk) and a component of $\partial_v(N(B_V))$ (a vertical disk). Hence, if D_c were a contact disk in $N(\hat{B})$, then its boundary would have to intersect a component of $\partial_v(N(B))$. It follows that the interior of D_c must enter into the region of $N(B)$ corresponding to a branch of ($S-$ the singular locus of B). But one can easily see from Fig. 7 that the complement of the singular locus of B in S is a connected surface. It follows that D_c has to intersect every I-fiber of B since D_c is transverse to I-fibers of B. In particular ∂D_c has to intersect both of the vertical annuli of $\partial_v(\hat{B}_v)$, which gives a contradiction.

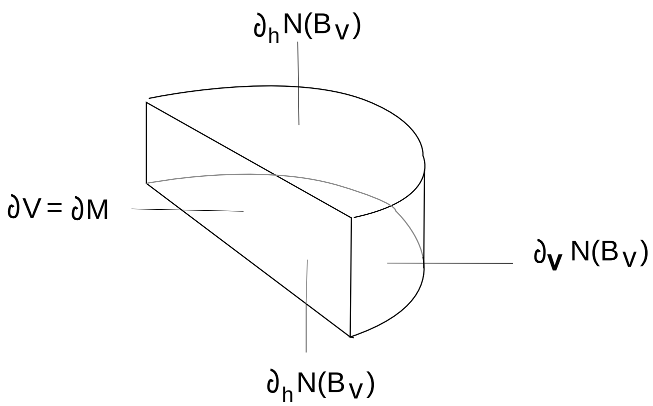
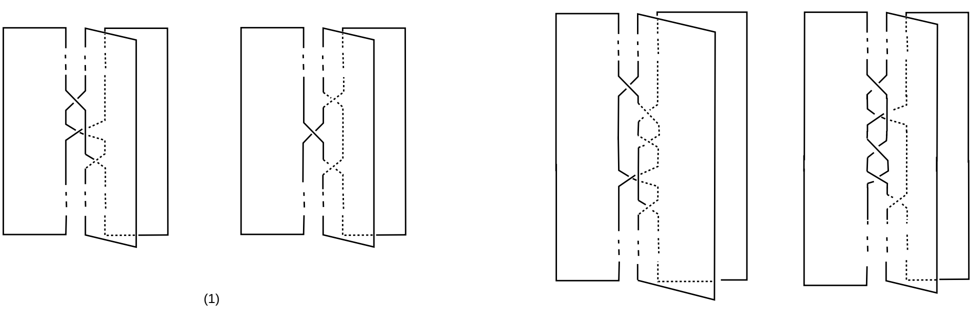


Fig. 10. A component of $V - \overset{\circ}{N}(B_v)$.

We now prove Proposition 3. By [18], the canonical Seifert surface of $\hat{\beta}$ has minimal genus. If the condition (1) of Proposition 3 holds, then the conclusion of Proposition 3 follows obviously from Lemma 5. Suppose that the condition (2) of Proposition 3 holds. To show that the 1-surgery on $\hat{\beta}$ gives a manifold with an essential lamination, we may assume, by Lemma 5, that η contains at most three syllabus, and they have different subscripts and all have power -1. But η contains at least two syllabuses. Suppose that the first and the second syllabuses of η are a_3^{-1} and a_2^{-1}, i.e. $\eta = a_3^{-1}a_2^{-1}\cdots$. Then since β is a shortest word, the word δ does not end with a syllabus in a_3. Suppose that δ ends with a syllabus in a_1. Then we have $\beta = \cdots a_1 a_3^{-1} a_2^{-1} \cdots$. By a band move isotopy of the Seifert surface as shown in Fig. 11(1) we get an isotopic 3-braid β' which contains two a_2^{-1}. (Algebraically, $\beta = \cdots a_1 a_3^{-1} a_2^{-1} \cdots = \cdots a_2^{-1} a_1 a_2^{-1} \cdots = \beta'$). Further the canonical Seifert surface of $\hat{\beta}'$ is isotopic to that of $\hat{\beta}$ and thus has minimal genus. So we may apply Lemma 5 to see that for the knot $\hat{\beta}' = \hat{\beta}$, the 1-surgery gives a manifold with essential lamination. Suppose then that δ ends with a syllabus in a_2. Since δ is assumed to contain at least two syllabuses, $\beta = \cdots a_1 a_2^k a_3^{-1} a_2^{-1} \cdots$. Again we may first use the band-isotopy as shown in Fig. 11(2) and then use the band isotopy of Fig. 11(1) to get an isotopic 3-braid whose canonical Seifert surface contains two bands corresponding to a_2^{-1}. Hence Proposition 3 follows from Lemma 5 in this case as well. Similarly one can treat the cases when η starts with $a_2^{-1}a_1^{-1}$ or with $a_1^{-1}a_3^{-1}$. The case when δ contains at most three syllabuses, each having power at most one, can be proved similarly. This proves Proposition 3 under its condition (2). Finally if the condition (3) of Proposition 3 holds then either condition (2) of Proposition 3 holds or one can get directly two letters a_i of the same subscript in δ and two letters a_j^{-1} of the same subscript in η. So again the Proposition follows from Lemma 5. □



(1) (2)

Fig. 11. The band move isotopies.

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